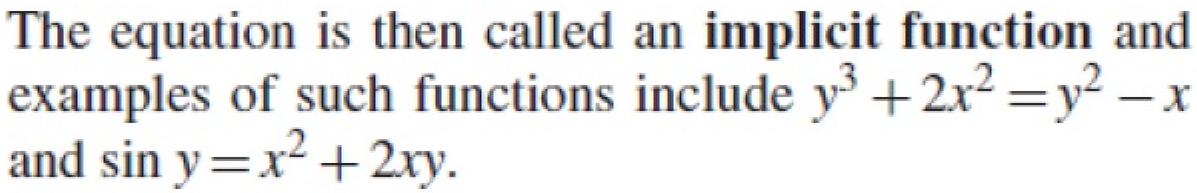
Implicit Functions

When an equation can be written in the form y = f(x) it is said to be an **explicit function**. Here are two examples of explicit functions:

$$y = 2x^3 - 3x + 4, \quad y = 2x \ln x$$

Sometimes it is impossible to make y the subject of the formula.



The equation is then called an **implicit function** and examples of such functions include $y^3 + 2x^2 = y^2 - x$ and $\sin y = x^2 + 2xy$.

BUT EVEN IMPLICIT EXPRESSIONS HAVE DERIVATIVES!

TO FIND THE DERIVATIVE WE USE THE CHAIN OR FUNCTION OF FUNCTION RULE

EXAMPLE

$$Y^2 = 6x^3$$

DIRECTLY $\Rightarrow Y = \sqrt{6x^3}$

$$= (6x^3)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(6x^3)^{-\frac{1}{2}} \times 18x^2$$

$$= \frac{9x^2}{\sqrt{6x^3}}$$

SOLVING USING IMPLICIT DIFFERENTIATION

$$Y^2 = 6x^3$$

$$2Y \cdot \frac{dy}{dx} = 18x^2$$

$$\frac{dy}{dx} = \frac{18x^2}{2Y}$$

$Y^2 = 6x^3$

$Y = \sqrt{6x^3}$

$$= \frac{9x^2}{Y}$$

$$= \frac{9x^2}{\sqrt{6x^3}}$$

Tutorial Worksheet – Differentiation 6

Implicit Functions

When an equation can be written in the form y = f(x) it is said to be an **explicit function**. Here are two examples of explicit functions:

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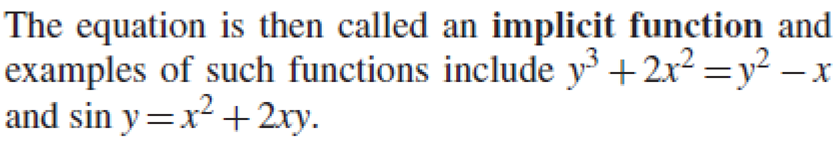
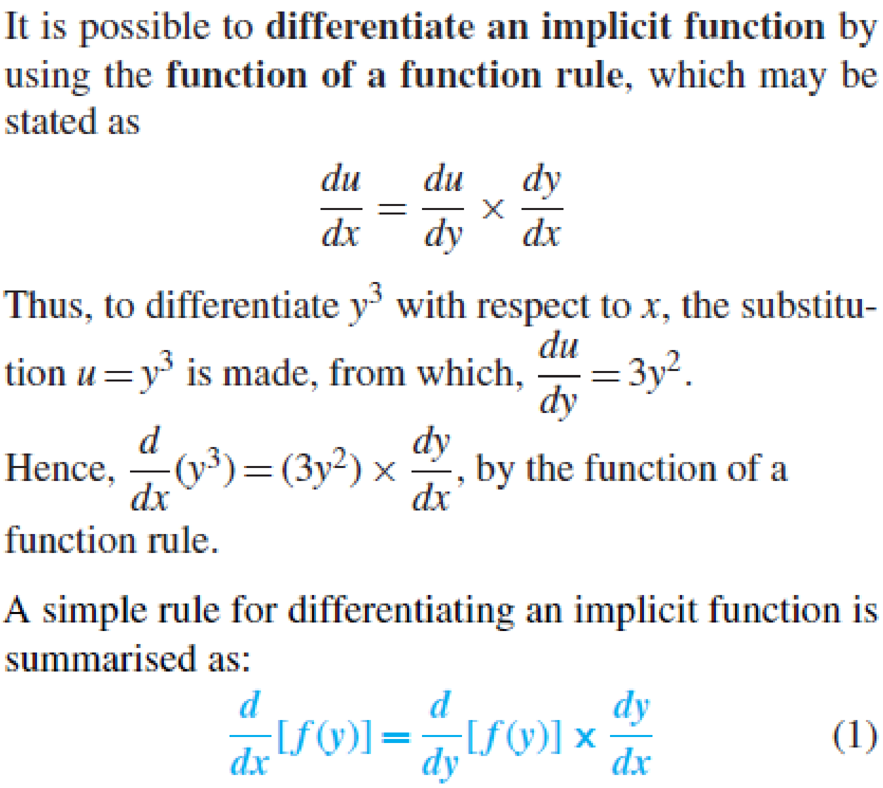


The equation is then called an **implicit function** and examples of such functions include $y^3 + 2x^2 = y^2 - x$ and $\sin y = x^2 + 2xy$.

Differentiation of Implicit Functions



It is possible to **differentiate an implicit function** by using the **function of a function rule**, which may be stated as

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Thus, to differentiate y^3 with respect to x, the substitution $u = y^3$ is made, from which, $\frac{du}{dy} = 3y^2$.

Hence, $\frac{d}{dx}(y^3) = (3y^2) \times \frac{dy}{dx}$, by the function of a function rule.

A simple rule for differentiating an implicit function is summarised as:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx} \qquad (1)$$

① DIFFERENTIATE THE FUNCTION $\boxed{U = 3y^4}$ WITH RESPECT TO $\underline{x}$

Remember

-

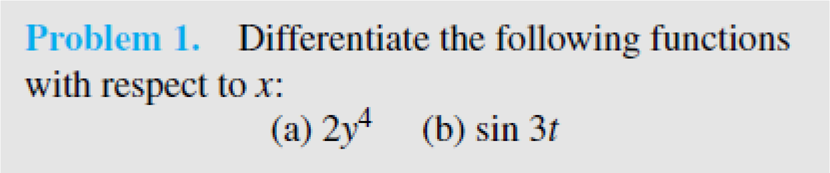
$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

① DIFFERENTIATE THE FUNCTION $\boxed{U = 3y^4}$

WITH RESPECT TO $\underline{x}$

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \boxed{12y^3 \times \frac{dy}{dx}}$$

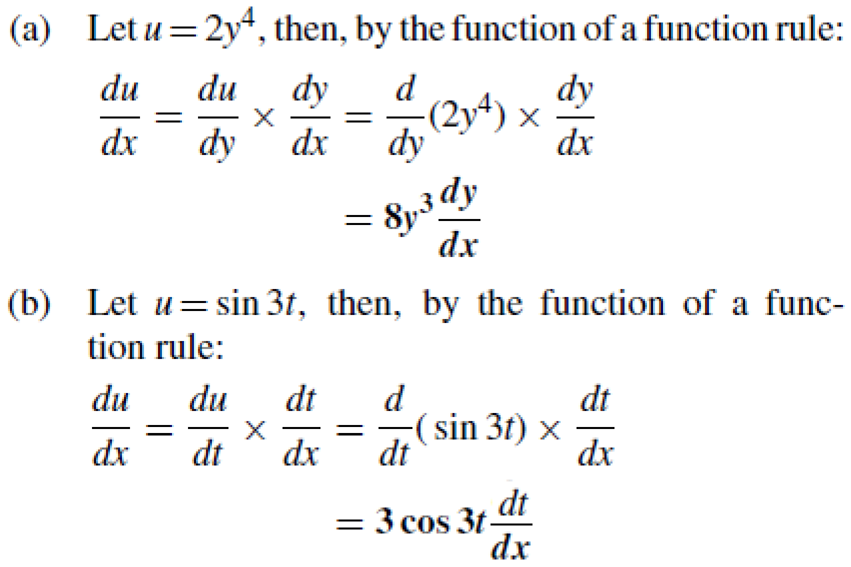
Sample Problem:



Problem 1. Differentiate the following functions with respect to x:

(a) $2y^4$ (b) $\sin 3t$

Sample Answer:



(a) Let $u = 2y^4$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(2y^4) \times \frac{dy}{dx}$$

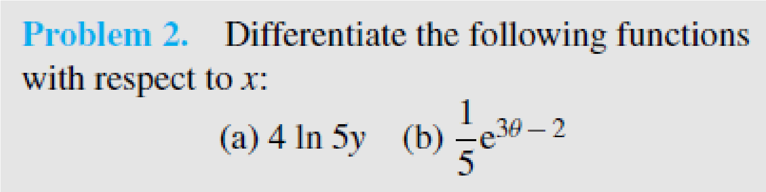
$$= 8y^3 \frac{dy}{dx}$$

(b) Let $u = \sin 3t$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx} = \frac{d}{dt}(\sin 3t) \times \frac{dt}{dx}$$

$$= 3\cos 3t \frac{dt}{dx}$$

Now you attempt to differentiate the following implicit functions....

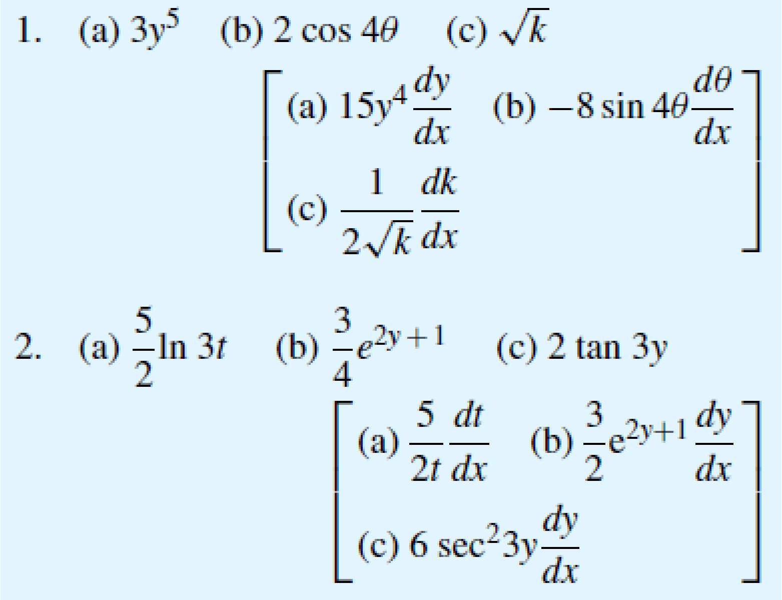


Problem 2. Differentiate the following functions with respect to x:

(a) $4 \ln 5y$ (b) $\frac{1}{5}e^{3\theta-2}$

Answers: (a) $\frac{4}{y}\frac{dy}{dx}$ (b) $\frac{3}{5}e^{3\theta-2}\frac{d\theta}{dx}$

Further practice problems using implicit differentiation…



1. (a) $3y^5$ (b) $2\cos 4\theta$ (c) $\sqrt{k}$

$\left[\text{(a) } 15y^4\frac{dy}{dx} \quad \text{(b) } -8\sin 4\theta\frac{d\theta}{dx} \quad \text{(c) } \frac{1}{2\sqrt{k}}\frac{dk}{dx}\right]$

2. (a) $\frac{5}{2}\ln 3t$ (b) $\frac{3}{4}e^{2y+1}$ (c) $2\tan 3y$

$\left[\text{(a) } \frac{5}{2t}\frac{dt}{dx} \quad \text{(b) } \frac{3}{2}e^{2y+1}\frac{dy}{dx} \quad \text{(c) } 6\sec^2 3y\frac{dy}{dx}\right]$

PROBLEM ②

(a) $4\ln 5y$

Let $u = 4\ln 5y \Rightarrow \frac{du}{dy} = \frac{4}{y}$

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

$$= \underline{\underline{\frac{4}{y} \cdot \frac{dy}{dx}}}$$

(b) ~~Let~~ $\frac{1}{5}e^{3\theta-2}$

Let $u = \frac{1}{5}e^{3\theta-2} \Rightarrow \frac{du}{d\theta} = \frac{3}{5}e^{3\theta-2}$

so $$\frac{du}{dx} = \frac{du}{d\theta} \times \frac{d\theta}{dx}$$

$$= \underline{\underline{\frac{3}{5}e^{3\theta-2} \times \frac{d\theta}{dx}}}$$

FURTHER PRACTICE PROBLEMS...

PROBLEM①:

(a) $3y^5$ Let $u = 3y^5 \therefore \frac{du}{dy} = 15y^4$

so $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \underline{\underline{15y^4 \times \frac{dy}{dx}}}$

(b) $2\cos 4\theta$ Let $u = 2\cos 4\theta \therefore \frac{du}{d\theta} = -8\sin 4\theta$

so $\frac{du}{dx} = \frac{du}{d\theta} \times \frac{d\theta}{dx} = \underline{\underline{-8\sin 4\theta \times \frac{d\theta}{dx}}}$

(c) $\sqrt{k}$ Let $u = \sqrt{k} = k^{\frac{1}{2}} \therefore \frac{du}{dk} = \frac{1}{2}k^{-\frac{1}{2}} = \frac{1}{2\sqrt{k}}$

so $\frac{du}{dx} = \frac{du}{dk} \times \frac{dk}{dx} = \underline{\underline{\frac{1}{2\sqrt{k}} \times \frac{dk}{dx}}}$

But how would you solve this one ??

② DIFFERENTIATE THE FUNCTION

$$U = 2x^5 + \underbrace{3y}_{}\,\underbrace{\cos 6x}_{}$$

WITH RESPECT TO $\underline{y}$

REMEMBER! $\boxed{\frac{du}{dy} = \frac{du}{dx} \times \frac{dx}{dy}}$

46.3 Differentiating implicit functions containing products and quotients

The product and quotient rules of differentiation must be applied when differentiating functions containing products and quotients of two variables.

For example, $\frac{d}{dx}(x^2y) = (x^2)\frac{d}{dx}(y) + (y)\frac{d}{dx}(x^2)$,

by the product rule

$$= (x^2)\left(1\frac{dy}{dx}\right) + y(2x),$$

by using equation(1)

$$= \mathbf{x^2\frac{dy}{dx} + 2xy}$$

Problem 3. Determine $\frac{d}{dx}(2x^3y^2)$

In the product rule of differentiation let $u = 2x^3$ and $v = y^2$.

Thus $\frac{d}{dx}(2x^3y^2) = (2x^3)\frac{d}{dx}(y^2) + (y^2)\frac{d}{dx}(2x^3)$

$$= (2x^3)\left(2y\frac{dy}{dx}\right) + (y^2)(6x^2)$$

$$= 4x^3y\frac{dy}{dx} + 6x^2y^2$$

$$= \mathbf{2x^2y\left(2x\frac{dy}{dx} + 3y\right)}$$

But how would you solve this one ??

② DIFFERENTIATE THE FUNCTION

$$U = 2x^5 + \underbrace{3y}\,\underbrace{\cos 6x}$$

WITH RESPECT TO $\underline{y}$

REMEMBER! $\boxed{\frac{du}{dy} = \frac{du}{dx} \times \frac{dx}{dy}}$

② DIFFERENTIATE THE FUNCTION

$$U = 2x^5 + \underbrace{3y}_{}\,\underbrace{\cos 6x}_{}$$

WITH RESPECT TO $\underline{Y}$

REMEMBER! $\boxed{\frac{du}{dy} = \frac{du}{dx} \times \frac{dx}{dy}}$

NOW LET US DO IT!!

$$= 10x^4 \times \frac{dx}{dy} + 3 \times \cos 6x + 3y \times (-6\sin 6x) \times \frac{dx}{dy}$$

$$\boxed{= 10x^4 \frac{dx}{dy} + 3\cos 6x - 18y \sin 6x \times \frac{dx}{dy}}$$

WHAT ABOUT THIS? USE PRODUCT RULE

$$\underset{f}{\uparrow}\!\!\!\!y^2 \underset{g}{\uparrow}\!\!\!\!x = \underset{\underset{f}{\uparrow}}{\underbrace{6x^4}}\underset{\underset{g}{\uparrow}}{y} + 3x^2$$

FIND $\frac{dy}{dx}$

$$2y \cdot \frac{dy}{dx} \cdot x + y^2 = 24x^3 \cdot y + \frac{dy}{dx} 6x^4 + 6x$$

$$\frac{dy}{dx}\left(2xy - 6x^4\right) = 24x^3y + 6x - y^2$$

$$\frac{dy}{dx} = \frac{24x^3y + 6x - y^2}{2xy - 6x^4}$$

③ GIVEN $-2y^3 + 3x^2 + 6xy = 12$

FIND $\frac{dy}{dx}$

③ GIVEN $-2y^3+3x^2+6xy=12$

FIND $\frac{dy}{dx}$ = ANSWER

ANSWER:

$$-6y^2 \cdot \frac{dy}{dx} + 6x + 6y + 6x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6x-6y^2) = -6x-6y$$

$$\frac{dy}{dx} = \frac{-6x-6y}{(6x-6y^2)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{6x+6y}{6y^2-6x}}$$

IMPLICIT DIFFERENTIATION

(A) DIFFERENTIATE THE FUNCTION WITH RESPECT TO Y

$$u = x^4 + 3x\cos 6Y$$

(USE CHAIN RULE) $\frac{du}{dY} = \frac{du}{dx} \times \frac{dx}{dY}$

ANSWER;

$$\frac{d(x^4+3x\cos 6Y)}{dY} = 4x^3 \times \frac{dx}{dY} + 3 \times \frac{dx}{dY}\cos 6Y + 3x(-\sin 6Y) \times 6$$

$$\therefore 4x^3\frac{dx}{dY} + 3\cos 6Y\frac{dx}{dY} - 18x\sin 6Y$$

(B) GIVEN $6Y^3 + 3x^4 - 5x^2Y + 2Y^2 = 8$

FIND $\frac{dY}{dx}$

$$18Y^2 \times \frac{dY}{dx} + 12x^3 - 10xY - 5x^2\frac{dY}{dx} + 4Y\frac{dY}{dx} = 0$$

$$\frac{dY}{dx}(18Y^2 - 5x^2 + 4Y) + 12x^3 - 10xY = 0$$

$$\frac{dY}{dx}(18Y^2 - 5x^2 + 4Y) = 10xY - 12x^3$$

$$\frac{dY}{dx} = \frac{10xY - 12x^3}{18Y^2 - 5x^2 + 4Y}$$

Now you attempt to differentiate the following implicit functions using the product and quotient rules....

Problem 4. Find $\frac{d}{dx}\left(\frac{3y}{2x}\right)$

Answer: $\frac{3}{2x^2}\left(x\frac{dy}{dx} - y\right)$

Problem 5. Differentiate $z = x^2 + 3x\cos 3y$ with respect to y.

Answer: $2x\frac{dx}{dy} - 9x\sin 3y + 3\cos 3y\frac{dx}{dy}$

Problem 6. Given $2y^2 - 5x^4 - 2 - 7y^3 = 0$, determine $\frac{dy}{dx}$

Answer: $\frac{20x^3}{(4y - 21y^2)}$

Problem 7. Determine the values of $\frac{dy}{dx}$ when $x = 4$ given that $x^2 + y^2 = 25$.

Answer: $\frac{dy}{dx} = -\frac{x}{y} = \pm\frac{4}{3}$

PROBLEM ④ FIND $\frac{d}{dx}\left(\frac{3Y}{2x}\right)$

USE QUOTIENT RULE. $u = 3Y$; $v = 2x$

so $\frac{d}{dx}\left(\frac{3Y}{2x}\right) = \frac{3 \times \frac{dY}{dx} \times 2x - 2 \times 3Y}{(2x)^2}$

$$= \frac{3 \times 2 \times \left[x\frac{dY}{dx} - Y\right]}{4x^2}$$

$$= \underline{\underline{\frac{3}{2x^2}\left[x\frac{dY}{dx} - Y\right]}}$$

PROBLEM ⑤ : DIFFERENTIATE WITH RESPECT TO $\underline{\underline{Y}}$ ⇒ $Z = x^2 + 3x \cos 3Y$

$$\frac{dZ}{dY} = 2x \times \frac{dx}{dY} + 3 \times \frac{dx}{dY} \times \cos 3Y - 9x \sin 3Y$$

PROBLEM ⑥ : GIVEN $2Y^2 - 5x^4 - 2 - 7Y^3 = 0$ FIND $\frac{dY}{dx}$

$$\Rightarrow 4Y \times \frac{dY}{dx} - 20x^3 - 0 - 21Y^2 \times \frac{dY}{dx} = 0$$

$$\Rightarrow \frac{dY}{dx}(4Y - 21Y^2) = 20x^3$$

$$\Rightarrow \underline{\underline{\frac{dY}{dx} = \frac{20x^3}{(4Y - 21Y^2)}}}$$

PROBLEM ②

FIND THE VALUE OF $\frac{dy}{dx}$ when $x = 4$ GIVEN THAT $x^2 + y^2 = 25$

FIRST DIFFERENTIATE IMPLICITLY!

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$$

$$= \frac{-x}{y}$$

NOW WHEN $x = 4 \Rightarrow$

$$(4)^2 + (y)^2 = 25$$

$$\Rightarrow y^2 = 25 - 16 = 9$$

$$\Rightarrow y = +/- 3$$

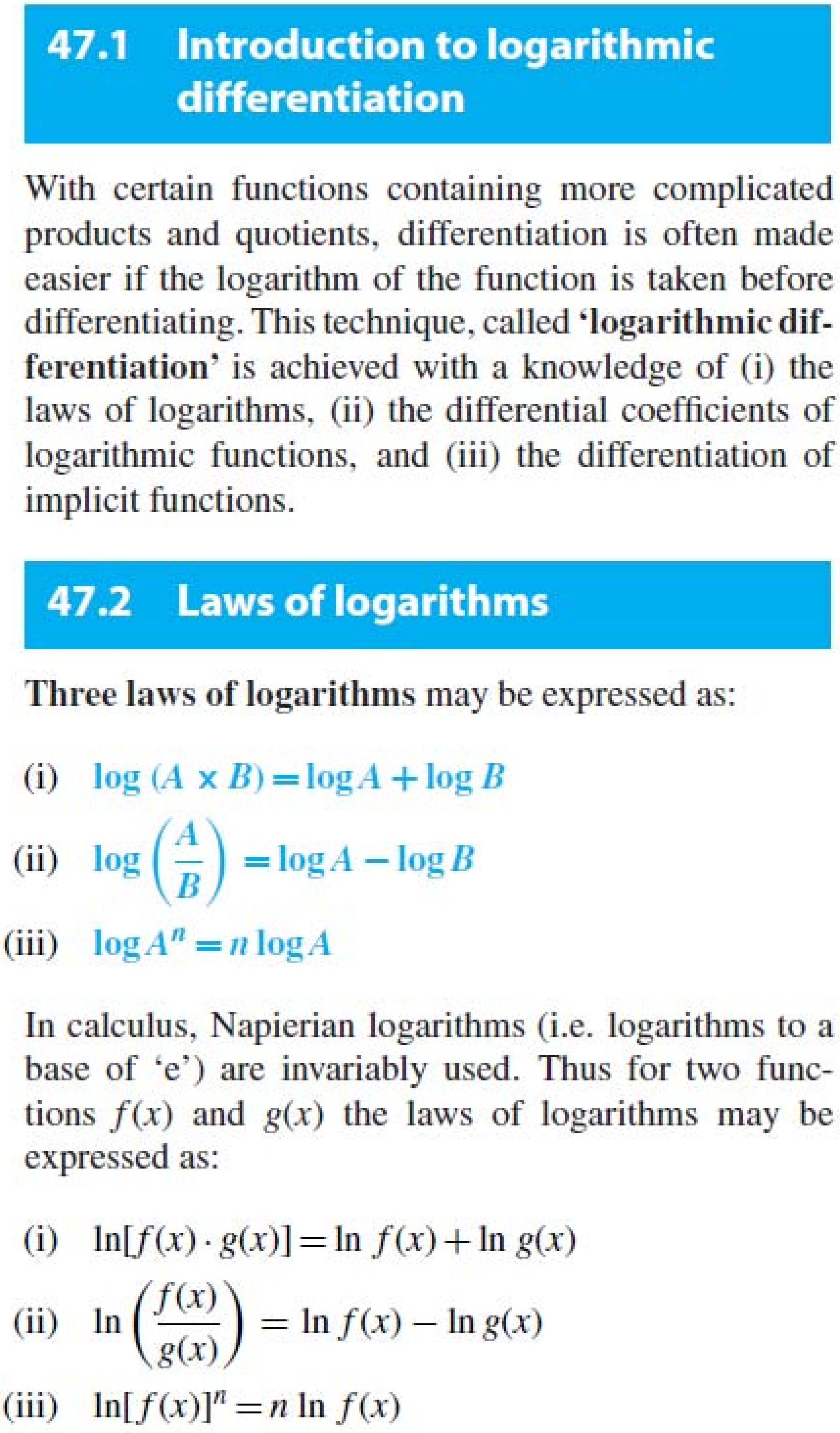
SO $\frac{dy}{dx} = \frac{-x}{y} = +/- \frac{4}{3}$

LOGARITHMIC DIFFERENTIATION

USE THE LAWS OF LOGS TO SIMPLIFY TAKING DERIVATIVE OF COMPLICATED EXPRESSIONS.

Tutorial Worksheet – Differentiation 7

Logarithmic Differentiation:



47.1 Introduction to logarithmic differentiation

With certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating. This technique, called **'logarithmic differentiation'** is achieved with a knowledge of (i) the laws of logarithms, (ii) the differential coefficients of logarithmic functions, and (iii) the differentiation of implicit functions.

47.2 Laws of logarithms

Three laws of logarithms may be expressed as:

(i) $\log(A \times B) = \log A + \log B$

(ii) $\log\left(\frac{A}{B}\right) = \log A - \log B$

(iii) $\log A^n = n \log A$

In calculus, Napierian logarithms (i.e. logarithms to a base of 'e') are invariably used. Thus for two functions $f(x)$ and $g(x)$ the laws of logarithms may be expressed as:

(i) $\ln[f(x) \cdot g(x)] = \ln f(x) + \ln g(x)$

(ii) $\ln\left(\frac{f(x)}{g(x)}\right) = \ln f(x) - \ln g(x)$

(iii) $\ln[f(x)]^n = n \ln f(x)$

SO LOGARITHMIC DIFFERENTIATION IS BASED AROUND THIS.

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

TRIVIAL EXAMPLE

$$Y = f(x) = x^3$$

STEP ① TAKE LOG ON BOTH SIDES

$$\ln Y = \ln(x^3)$$

$$\ln Y = 3\ln x$$

STEP ② TAKE THE DERIVATIVE ON BOTH SIDES

$$\frac{1}{Y} \times \frac{dY}{dx} = 3 \times \frac{1}{x}$$

$$\Rightarrow \frac{dY}{dx} = \frac{3Y}{x}$$

$$= \frac{3\cancel{x}^{\cancel{3}}}{\cancel{x}}$$

$$= \underline{3x^2}$$

What about this

$$Y = 3^x$$

WE USE LOGS:

STEP ① TAKE LOGS

$$\ln Y = \ln(3^x)$$

$$\ln Y = x\ln 3$$

STEP ② TAKE DERIVATIVE

$$\frac{1}{Y} \cdot \frac{dy}{dx} = \ln 3 + 0$$

$$= \ln 3$$

STEP ③ SUBSTITUTE BACK $Y=3^x$

$$\frac{dy}{dx} = Y\ln 3$$

$$= 3^x \ln 3$$

$$\boxed{\frac{dy}{dx} = \ln 3\, 3^x}$$

$$Y = x^x$$

$$\boxed{\begin{aligned} f(x) &= g(x)\,h(x) \\ f'(x) &= g'h + h'g \end{aligned}}$$

$$\ln Y = \ln(x^x)$$

$$\ln Y = x\ln x$$

$$\frac{1}{Y}\frac{dy}{dx} = \ln x + \frac{1}{x}x$$

$$= \ln x + 1$$

$$\frac{dy}{dx} = Y(\ln x + 1)$$

$$= x^x(\ln x + 1)$$

$$= x^x \ln x + x^x$$

$$\boxed{\frac{dy}{dx} = x^x \ln x + x^x}$$

LOGARITHMIC DIFFERENTIAT

① $Y = 3x^{(2x^3-5)}$

$\ln(AB)$
$= \ln A + \ln B$

TAKE LOGS!

$$\ln Y = \ln\left(3x^{(2x^3-5)}\right)$$

$$= \ln 3 + \ln x^{(2x^3-5)}$$

$$= \ln 3 + (2x^3-5) \cdot \ln x$$

$$= \ln 3 + 2x^3 \ln x - 5 \ln x$$

NOW DIFFERENTIATE!

$$\frac{1}{Y} \times \frac{dY}{dx} = 0 + 6x^2 \ln x + 2x^{\cancel{3}} \times \frac{1}{\cancel{x}} - \frac{5}{x}$$

$$\frac{1}{Y} \times \frac{dY}{dx} = 6x^2 \ln x + 2x^2 - \frac{5}{x}$$

$$\therefore \frac{dY}{dx} = Y \times \left[6x^2 \ln x + 2x^2 - \frac{5}{x}\right]$$

$$\boxed{= 3x^{(2x^3-5)} \times \left[6x^2 \ln x + 2x^2 - \frac{5}{x}\right].}$$

Now you try this one ...

DIFFERENTIATE USING LOGARITHMIC DIFFERENTIATION THE FOLLOWING:

$$\boxed{Y = 3x^{(10-6x^2)}}$$

Remember the key steps...

Step 1: Take Logs on both sides

Step 2: Simplify the Log expression as much as possible

Step 3: Perform the derivative

Step 4: Gather terms to write as final dy/dx expression

LOGARITHMIC DIFFERENTIATION

DIFFERENTIATE USING LOGARITHMIC DIFFERENTIATION THE FOLLOWING:

$$\boxed{Y = 3x^{(10-6x^2)}}$$

TAKE LOGS ON BOTH SIDES:

$$\ln Y = \ln\left(3x^{(10-6x^2)}\right)$$

$$= \ln 3 + \ln x^{(10-6x^2)}$$

$$= \ln 3 + (10-6x^2)\ln x$$

$$= \ln 3 + 10\ln x - 6x^2\ln x$$

NOW DIFFERENTIATE:

$$\frac{1}{Y}\cdot\frac{dY}{dx} = 0 + \frac{10}{x} - 12x\ln x - 6x^{\cancel{2}} \times \frac{1}{\cancel{x}}$$

$$\frac{1}{Y}\frac{dY}{dx} = \frac{10}{x} - 12x\ln x - 6x$$

$$\frac{dY}{dx} = Y \times \left[\frac{10}{x} - 12x\ln x - 6x\right]$$

$$= 3x^{(10-6x^2)} \times \left[\frac{10}{x} - 12x\ln x - 6x\right]$$

NOW CONSIDER THIS EXAMPLE...

$$Y = \frac{(x-3)^4 \times (x^2+3x)^2}{(x+5)^6}$$

FIND $\frac{dy}{dx}$!! (NOTE, ONLY INTERESTED IN CORRECT ANSWER BUT IT DOES NOT NEED TO BE SIMPLIFIED)

ANSWER

$$Y = \frac{(x-3)^4 (x^2+3x)^2}{(x+5)^6}$$

$$Y = \frac{AB}{C}$$

$$\ln Y = \ln A + \ln B - \ln C$$

STEP ①

TAKE LOGS

$$\ln Y = \ln\left(\text{ALL THIS STUFF!}\right)$$

AND SIMPLIFY DOWN

$$= \ln(x-3)^4 + \ln(x^2+3x)^2 - \ln(x+5)^6$$

$$= \underline{4\ln(x-3)} + \underline{2\ln(x^2+3x)} - \underline{6\ln(x+5)}$$

STEP ②

TAKE DERIVATIVE

$$\frac{1}{Y}\frac{dy}{dx} = \frac{4}{x-3} + \frac{1}{x^2+3x} \times 2 \times (2x+3) - \frac{1}{x+5} \times 6$$

$$= \frac{4}{x-3} + \frac{4x+6}{x^2+3x} - \frac{6}{x+5}$$

FINALLY

$$\frac{dy}{dx} = Y \times \left(\frac{4}{x-3} + \frac{4x+6}{x^2+3x} - \frac{6}{x+5}\right)$$

$$= \frac{(x-3)^4 (x^2+3x)^2}{(x+5)^6} \times \left(\frac{4}{x-3} + \frac{4x+6}{x^2+3x} - \frac{6}{x+5}\right)$$

Try for Homework this one !!

FIND DERIVATIVE USING LOGARITHMIC DIFFERENTIATION OF THE FOLLOWING...

$$Y = \frac{\sin(6x^2-2)\, e^{4x^2}}{(x^2-2x+3)^3 \times \cos(3x)}$$

FIND DERIVATIVE USING LOGARITHMIC DIFFERENTIATION OF THE FOLLOWING...

$$Y = \frac{\sin(6x^2-2)\, e^{4x^2}}{(x^2-2x+3)^3 \times \cos(3x)}$$

STEP ①
TAKE LOGS

$$\ln Y = \ln(\sin(6x^2-2)) + \ln e^{4x^2} - \ln(x^2-2x+3)^3 - \ln(\cos(3x))$$

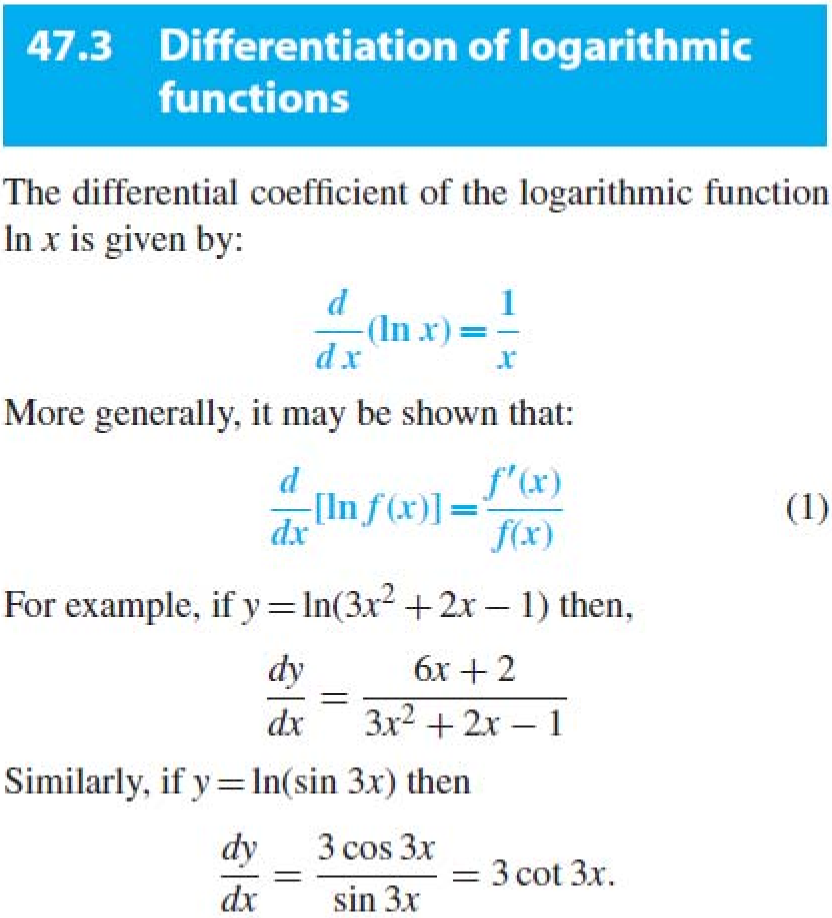
$$= \underline{\ln(\sin(6x^2-2))} + \underline{4x^2} - \underline{3\ln(x^2-2x+3)} - \underline{\ln(\cos(3x))}$$

STEP ②
TAKE DERIVATIVE

$$\frac{1}{Y}\frac{dY}{dx} = \frac{1}{\sin(6x^2-2)} \times \cos(6x^2-2) \times 12x + 8x - \frac{1}{(x^2-2x+3)} \times 3 \times (2x-2)$$

$$- \frac{1}{\cos(3x)} \times -\sin(3x) \times 3$$

$$\therefore \frac{dY}{dx} = Y \times \left(\frac{12x\cos(6x^2-2)}{\sin(6x^2-2)} + 8x - \frac{6x-6}{x^2-2x+3} + \frac{3\sin(3x)}{\cos(3x)} \right)$$



47.3 Differentiation of logarithmic functions

The differential coefficient of the logarithmic function ln x is given by:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

More generally, it may be shown that:

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)} \qquad (1)$$

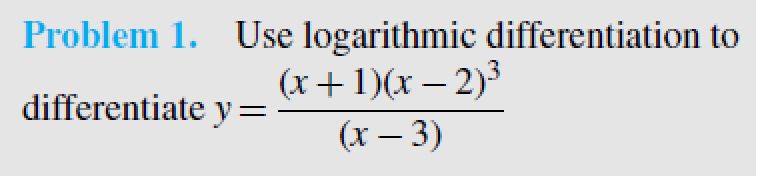
For example, if $y = \ln(3x^2 + 2x - 1)$ then,

$$\frac{dy}{dx} = \frac{6x + 2}{3x^2 + 2x - 1}$$

Similarly, if $y = \ln(\sin 3x)$ then

$$\frac{dy}{dx} = \frac{3\cos 3x}{\sin 3x} = 3\cot 3x.$$

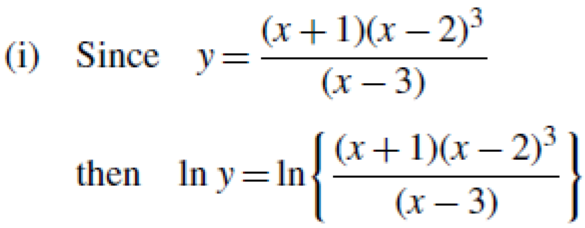
Sample Problem:



Problem 1. Use logarithmic differentiation to differentiate $y = \dfrac{(x+1)(x-2)^3}{(x-3)}$

Sample Answer:

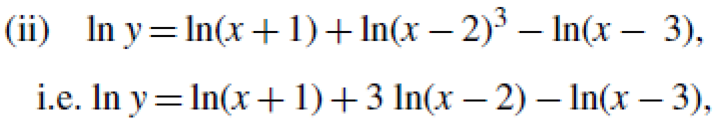
The answer is quite complex but if we tackle it step by step it is not so hard to understand!



(i) Since $y = \dfrac{(x+1)(x-2)^3}{(x-3)}$

then $\ln y = \ln\left\{\dfrac{(x+1)(x-2)^3}{(x-3)}\right\}$

Now use our Laws of Logarithms to re-write the expression..



(ii) $\ln y = \ln(x+1) + \ln(x-2)^3 - \ln(x- \ 3)$,

i.e. $\ln y = \ln(x+1) + 3\ln(x-2) - \ln(x-3)$, Now it is time to differentiate and apply some algebra..

(iii) Differentiating with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}$$

(iv) Rearranging gives:

$$\frac{dy}{dx} = y\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}$$

(v) Substituting for y gives:

$$\frac{dy}{dx} = \frac{(x+1)(x-2)^3}{(x-3)}\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}$$

Now you attempt to use logarithmic differentiation on the following:

Problem 2. Differentiate $y = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$ with respect to x and evalualte $\frac{dy}{dx}$ when $x = 3$.

Answer:

$$\frac{dy}{dx} = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\left\{\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right\}$$

Problem 4. Differentiate $y = \frac{x^3 \ln 2x}{e^x \sin x}$ with respect to x

Answer:

$$\frac{dy}{dx} = \frac{x^3 \ln 2x}{e^x \sin x}\left\{\frac{3}{x} + \frac{1}{x \ln 2x} - 1 - \cot x\right\}$$

PROBLEM ② $Y = \dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$

FIRST TAKE LOGS $\Rightarrow$

$$\operatorname{Ln} Y = \operatorname{Ln}\left[\frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\right]$$

$$\Rightarrow \operatorname{Ln} Y = \operatorname{Ln}(x-2)^{3/2} - \operatorname{Ln}(x+1)^2 - \operatorname{Ln}(2x-1)$$

$$= \tfrac{3}{2}\operatorname{Ln}(x-2) - 2\operatorname{Ln}(x+1) - \operatorname{Ln}(2x-1)$$

NOW TAKE DERIVATIVES.

$$\frac{1}{Y} \times \frac{dy}{dx} = \frac{3}{2} \times \frac{1}{(x-2)} - \frac{2}{(x+1)} - \frac{1 \times 2}{(2x-1)}$$

$$\Rightarrow \frac{dY}{dx} = Y \times \left[\frac{3}{2}\frac{1}{(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right]$$

$$\Rightarrow \frac{dY}{dx} = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)} \times \left[\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right]$$

PROBLEM ④ $Y = \dfrac{x^3 \operatorname{Ln} 2x}{e^x \sin x}$

FIRST TAKE LOGS $\Rightarrow \operatorname{Ln} Y = \operatorname{Ln}\left[\dfrac{x^3 \operatorname{Ln} 2x}{e^x \sin x}\right]$

$$\Rightarrow \operatorname{Ln} Y = \operatorname{Ln} x^3 + \operatorname{Ln}[\operatorname{Ln} 2x] - \operatorname{Ln} e^x - \operatorname{Ln} \sin x$$

$$= 3 \operatorname{Ln} x + \operatorname{Ln}[\operatorname{Ln} 2x] - x - \operatorname{Ln}(\sin x)$$

NOTE $\operatorname{Ln} e^x$
$= x \operatorname{Ln}(e)$
$= x$

NOW TAKE DERIVATIVES:

$$\frac{1}{Y} \times \frac{dy}{dx} = \left[\frac{3}{x} + \frac{1}{\operatorname{Ln}(2x)} \times \frac{1}{(\cancel{2}x)} \times \frac{\cancel{2}}{1} - \cancel{1} - \frac{1}{\sin x} \times \cos x\right]$$

$$= \left[\frac{3}{x} + \frac{1}{x \operatorname{Ln}(2x)} - 1 - \frac{\cos x}{\sin x}\right]$$

$$\therefore \frac{dy}{dx} = Y \times \left[\frac{3}{x} + \frac{1}{x \operatorname{Ln}(2x)} - 1 - \cot x\right]$$

$$\therefore \frac{dy}{dx} = \frac{x^3 \operatorname{Ln}(2x)}{e^x \sin x} \times \left[\frac{3}{x} + \frac{1}{x \operatorname{Ln}(2x)} - 1 - \cot x\right]$$

47.4 Differentiation of $[f(x)]^x$

Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then logarithmic differentiation must be used.

Problem 5. Determine $\frac{dy}{dx}$ given $y = x^x$.

Taking Napierian logarithms of both sides of $y = x^x$ gives:

$\ln y = \ln x^x = x \ln x$, by law (iii) of Section 47.2

Differentiating both sides with respect to x gives:

$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{x}\right) + (\ln x)(1)$, using the product rule

$$\text{i.e.} \quad \frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\text{from which,} \quad \frac{dy}{dx} = y(1 + \ln x)$$

$$\text{i.e.} \quad \boldsymbol{\frac{dy}{dx} = x^x(1 + \ln x)}$$

Now you attempt to use logarithmic differentiation on the following:

Problem 6. Evaluate $\frac{dy}{dx}$ when $y = (x + 2)^x$

Answer

$$\frac{dy}{dx} = (x + 2)^x \left\{ \frac{x}{x + 2} + \ln(x + 2) \right\}$$

Problem 8. Differentiate x^{3x+2} with respect to x.

Answer

$$\frac{dy}{dx} = x^{3x+2} \left\{ 3 + \frac{2}{x} + 3 \ln x \right\}$$

PROBLEM ⑥

$$Y = (x+2)^x$$

FIRST TAKE LOGS $\Rightarrow$ $\ln Y = \ln (x+2)^x$

$\Rightarrow \ln Y = x \ln (x+2)$

NOW TAKE DERIVATIVES:

$$\frac{1}{Y} \times \frac{dY}{dx} = 1 \times \ln(x+2) + \frac{1}{(x+2)} \times x \quad \left(\text{PRODUCT RULE}\right)$$

$$= \ln(x+2) + \frac{x}{x+2}$$

$$\therefore \frac{dY}{dx} = Y \times \left[\ln(x+2) + \frac{x}{x+2}\right]$$

$$= (x+2)^x \times \left[\frac{x}{x+2} + \ln(x+2)\right]$$

PROBLEM ⑧

$$Y = x^{3x+2}$$

FIRST TAKE LOGS $\Rightarrow$ $\ln Y = \ln x^{3x+2}$

$\Rightarrow \ln Y = (3x+2) \ln x$

NOW TAKE DERIVATIVE:

$$\frac{1}{Y} \times \frac{dY}{dx} = 3 \times \ln x + \frac{(3x+2)}{x} \quad \left(\text{PRODUCT RULE}\right)$$

$$\therefore \frac{dY}{dx} = Y \times \left[3 \ln x + \frac{3x+2}{x}\right]$$

$$= Y \times \left[3 \ln x + \frac{3x}{x} + \frac{2}{x}\right]$$

$$= x^{3x+2}\left[3 + \frac{2}{x} + 3 \ln x\right]$$

Tutorial Worksheet – Integrals 7

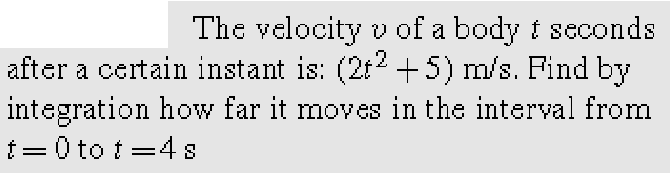
Integration areas and other relationship variables:

The area under a curve can also be used to determine other related variables.

Motion:

Area under a velocity / time graph gives displacement.
Area under an acceleration / time graph gives velocity.

Sample Problem 1:



The velocity v of a body t seconds after a certain instant is: $(2t^2+5)$ m/s. Find by integration how far it moves in the interval from $t=0$ to $t=4$ s

Sample Answer 1:

Since $2t^2+5$ is a quadratic expression, the curve $v=2t^2+5$ is a parabola cutting the v-axis at $v=5$, as shown in Fig. 55.4.

The distance travelled is given by the area under the v/t curve (shown shaded in Fig. 55.4).

By integration,

$$\text{shaded area} = \int_0^4 v\,dt$$

$$= \int_0^4 (2t^2+5)\,dt$$

$$= \left[\frac{2t^3}{3}+5t\right]_0^4$$

$$= \left(\frac{2(4^3)}{3}+5(4)\right)-(0)$$

i.e. **distance travelled = 62.67 m**

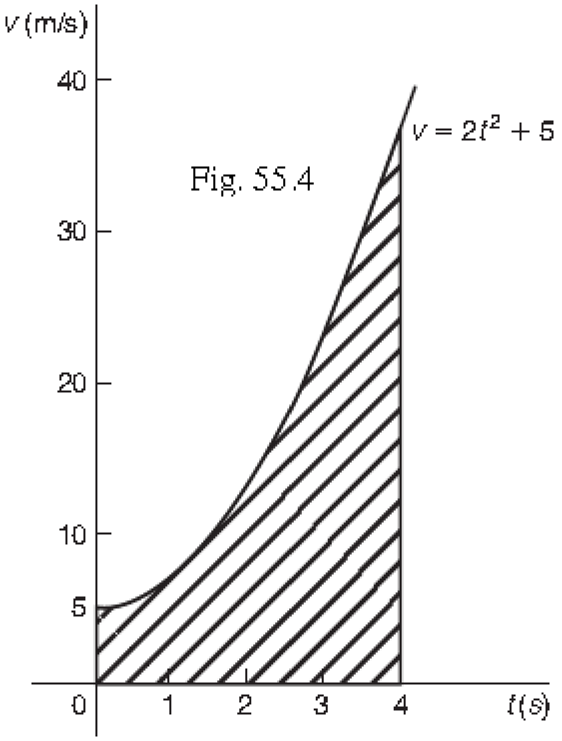
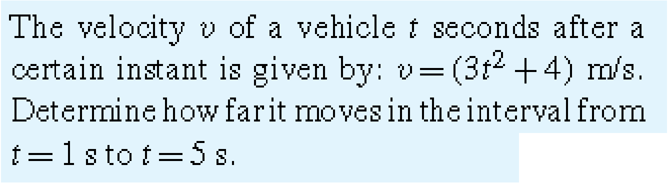


Fig. 55.4

Now you attempt this problem..



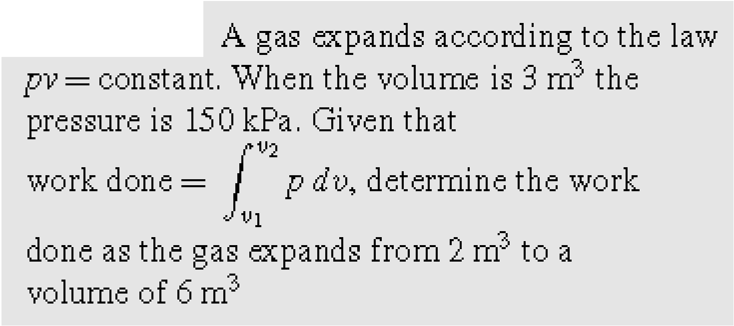
The velocity v of a vehicle t seconds after a certain instant is given by: $v=(3t^2+4)$ m/s. Determine how far it moves in the interval from $t=1$ s to $t=5$ s.

ANSWER: 140 m

Work:

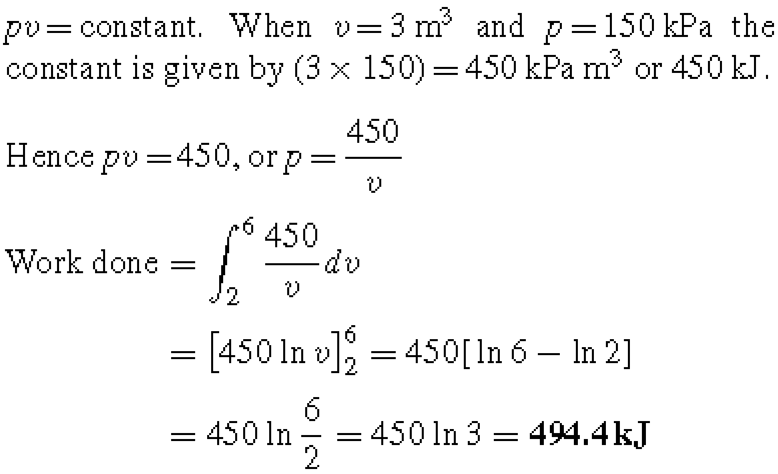
Area under a force / distance graph gives work done.
Area under a gas expansion graph of pressure / volume gives work done.

Sample Problem 2:



A gas expands according to the law $pv =$ constant. When the volume is 3 m^3 the pressure is 150 kPa. Given that work done $= \int_{v_1}^{v_2} p\,dv$, determine the work done as the gas expands from 2 m^3 to a volume of 6 m^3

Sample Answer 2:



$pv =$ constant. When $v = 3\ m^3$ and $p = 150$ kPa the constant is given by $(3 \times 150) = 450$ kPa m^3 or 450 kJ.

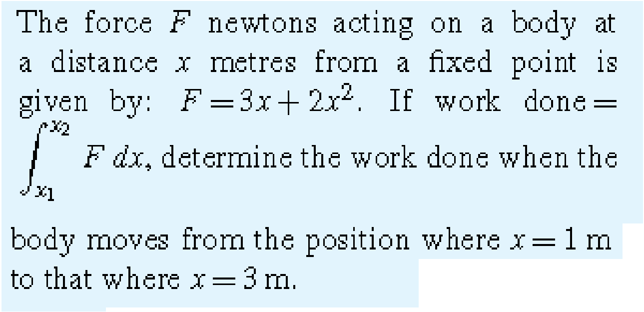
Hence $pv = 450$, or $p = \frac{450}{v}$

$$\text{Work done} = \int_2^6 \frac{450}{v}\,dv$$

$$= [450 \ln v]_2^6 = 450[\ln 6 - \ln 2]$$

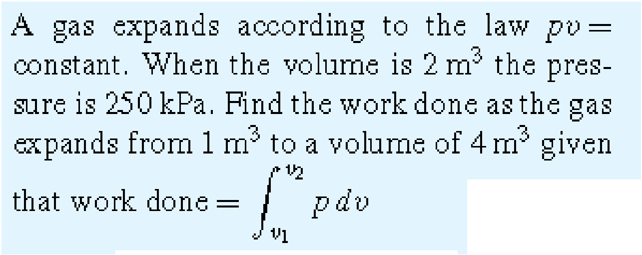
$$= 450 \ln \frac{6}{2} = 450 \ln 3 = \mathbf{494.4\,kJ}$$

Now you attempt these problems using the stated relationships for work done.



The force F newtons acting on a body at a distance x metres from a fixed point is given by: $F = 3x + 2x^2$. If work done $= \int_{x_1}^{x_2} F\,dx$, determine the work done when the body moves from the position where $x = 1$ m to that where $x = 3$ m.

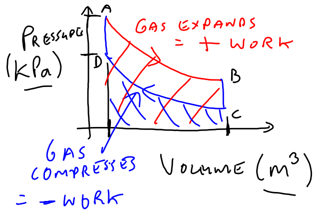
ANSWER: 29.33 Nm



A gas expands according to the law $pv =$ constant. When the volume is 2 m^3 the pressure is 250 kPa. Find the work done as the gas expands from 1 m^3 to a volume of 4 m^3 given that work done $= \int_{v_1}^{v_2} p\,dv$

ANSWER: 693.1 kJ

APPLICATION OF INTEGRALS IN GAS EXPANSION



NET WORK = RED AREA − BLUE AREA

GAS LAW IS $\boxed{PV = \text{CONSTANT}}$

AT Ⓐ PRESSURE = 600kPa
VOLUME = $1 m^3$

AT Ⓑ VOLUME = $6 m^3$
PRESSURE = 100kPa

FIND WORK DONE
IN GAS GOING FROM Ⓐ to Ⓑ

ANSWER $PV = \text{CONSTANT} = 600$

$PV = 600 \therefore \boxed{P = \frac{600}{V}}$

BY DEFINITION:

$$\text{WORK} = \int_{V_1}^{V_2} P\,dV$$

$$= \int_1^6 \frac{600}{V}\,dV$$

$$= \left[600 \ln V\right]_1^6$$

$$= 600 \times \ln(6) - 600 \times \ln(1)$$

$$= 1{,}075\,\text{kJoules}$$

WHAT IS WORK DONE FROM
Ⓒ to Ⓓ

AT Ⓒ PRESSURE = 20kPa
VOLUME = $6 m^3$

IN THIS CASE! $PV = \text{CONSTANT}$

NOW $20 \times 6 = 120$

$\boxed{PV = 120}$

HENCE AT POINT Ⓓ
(VOLUME = $1 m^3$)
PRESSURE = 120 kPa

LET US FIND WORK (NEGATIVE)
FROM Ⓒ to Ⓓ

$$\text{WORK} = \int_6^1 \frac{120}{V}\,dV$$

$$= \left[120 \ln V\right]_6^1$$

$$= 0 - 215\,\text{k Joules}$$

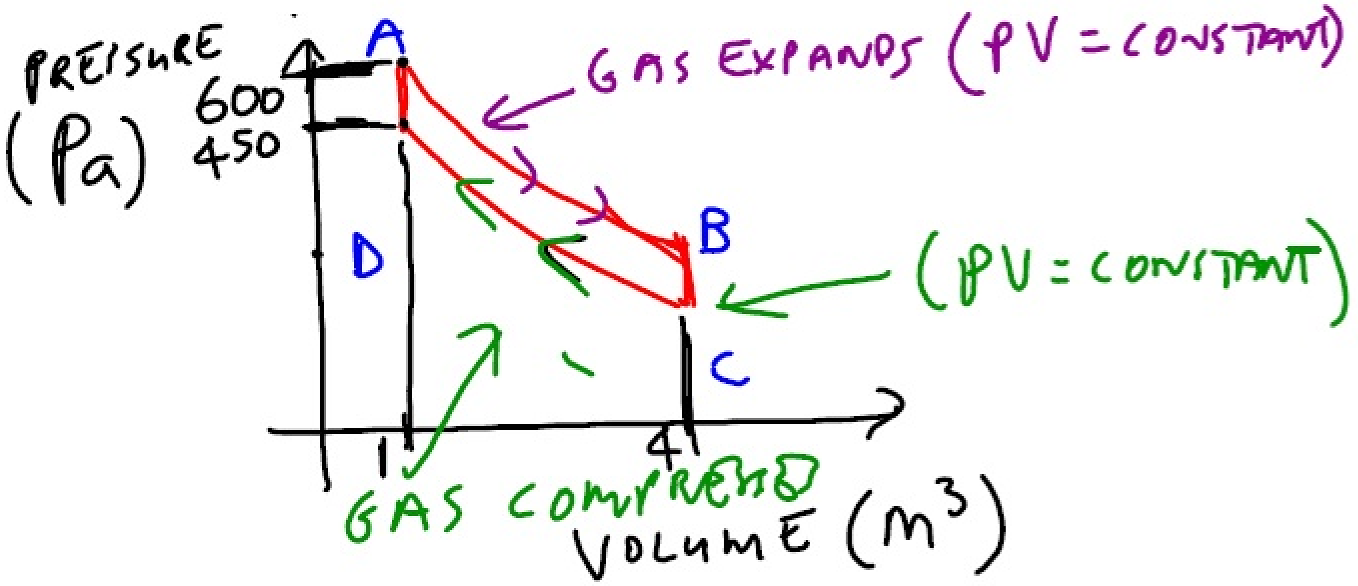
$$= -215\,\text{k Joules}$$

FINALLY

NET WORK = RED − BLUE
= 860 k Joules

Now you try one...

Find **TOTAL** work done by the gas for cycle A-B-C-D-A



Remember the following key points:

Formula for Gas is: PV = Constant

$$\text{WORK} = \int_{V_i}^{V_f} P\, dU$$

From A->B the Work done is **Positive**

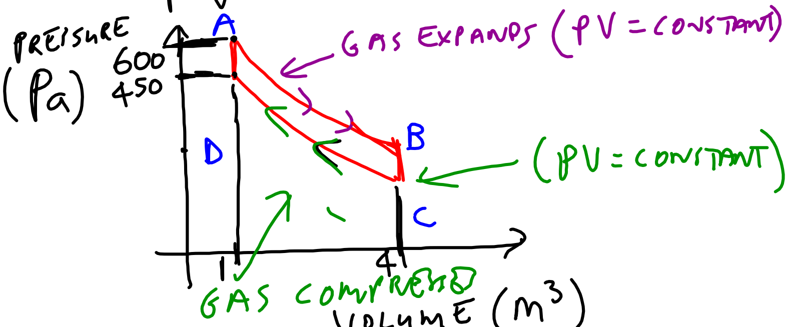
From C->D the Work done is **Negative**

At A: P = 600, V=1
At D: P = 450 V =1

At B and C: V = 4

THERMODYNAMIC CYCLE FOR A GAS EXPANDS ACCORDING TO THIS

PV = CONSTANT



CALCULATE USING CALCULUS NET WORK DONE

ANSWER: WORK IS THE AREA UNDER THE PV CURVE FOR THE GAS

$$\text{WORK} = \int_{V_i}^{V_f} P\, dV$$

PV = CONSTANT = 600

$\therefore P = \frac{600}{V}$

GAS EXPANDS DOES POSITIVE "+" WORK

GAS COMPRESSED DOES NEGATIVE "−" WORK

NOW WE USE INTEGRATION TO FIND WORK

Ⓐ → Ⓑ $\text{WORK} = \int_1^4 ??\, dV$

$$= \int_1^4 \frac{600}{V} dV = \Big[600 \ln V\Big]_1^4$$

$$= 831 \text{ JOULES}$$

Ⓒ → Ⓓ

$$\text{WORK} = \int_4^1 \frac{450}{V} dV = \Big[450 \ln V\Big]_4^1$$

$$= -623 \text{ JOULES}$$

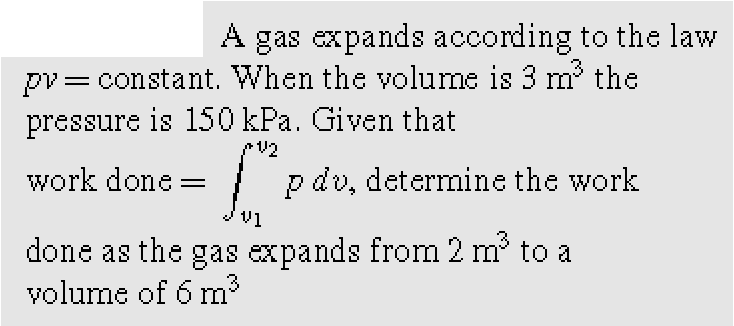
FINALLY TOTAL WORK $= 831 - 623$

$$= 208 \text{ JOULES}$$

Work:

Area under a force / distance graph gives work done.
Area under a gas expansion graph of pressure / volume gives work done.

Sample Problem 2:



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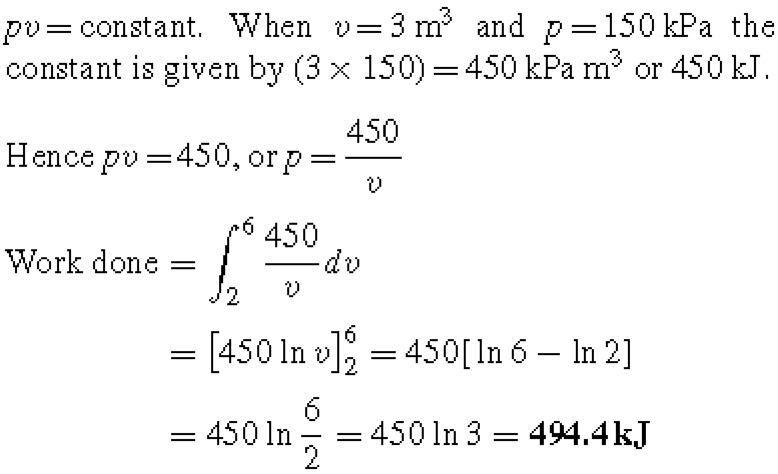
Sample Answer 2:



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$$= [450\ln v]_2^6 = 450[\ln 6 - \ln 2]$$
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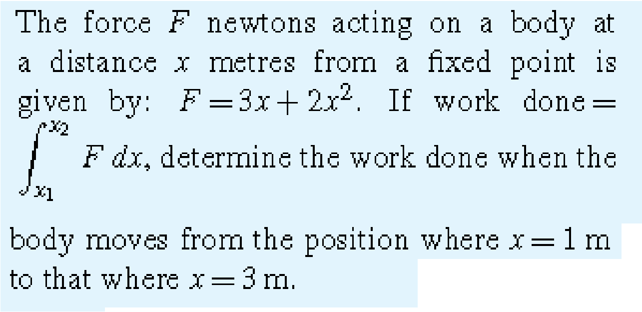

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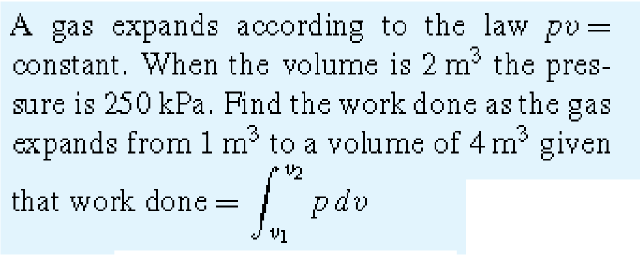
ANSWER: 29.33 Nm



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ANSWER: 693.1 kJ

INTEGRALS 7

$V = (3t^2 + 4)$ m/s

$t = 1s$, to $t = 5s$

DISTANCE $= \int v \, dt$

$= \int_1^5 (3t^2 + 4) \, dt$

$= \left[t^3 + 4t \right]_1^5$

$= [145 - 5] = \underline{140m}$

1.5

13.5 = 1.5

$F = 3x + 2x^2$

WORK $= \int F \, dx$

$= \int_1^3 (3x + 2x^2) \, dx$

$= \left[\cancel{\frac{3x^2}{3} + \frac{2x^3}{3}} \right] = \left[\frac{3x^2}{2} + \frac{2x^3}{3} \right]_1^3$

$= [31.5 - 2.16666]$

$= \underline{29.3333 \text{ Nm}}$

GAS EXPANDS ACCORDING TO.

$PV = \text{Constant}$

When ~~P~~ $V = 2$ $\quad P = 250\ \text{kPa}$

$\therefore 2 \times 250\text{k} = \text{Constant}$

$\Rightarrow \text{Constant} = 500\text{k}.$

So $PV = 500\text{k}.$

$$\text{WORK} = \int_{V_1}^{V_2} P\,dV$$

But $PV = 500\text{k}$ $\therefore P = \frac{500\text{k}}{V}$

$$\text{So WORK} = 500\text{k} \int_{V_1}^{V_2} \frac{1}{V}\,dV$$

$$= 500\text{k} \left[\ln V \right]_1^4$$

$$= 500\text{k} \times \left[\ln(4) - \ln(1) \right]$$

$$= 500\text{k} \times \left[1.3862 - 0 \right]$$

$$= \underline{\underline{693.1\ \text{kJ}}}$$